Carnot's Theorem and the Japanese Theorem

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I present a proof of Carnot's theorem as well as a famous consequence—the Japanese theorem.

Theorem 1 (Carnot's Theorem). In $\triangle ABC$, let M, N, and P be the midpoints of BC, CA, and AB. Let O be the circumcenter. Then

$$OM + ON + OP = R + r,$$

where R is the circumradius and r is the inradius.

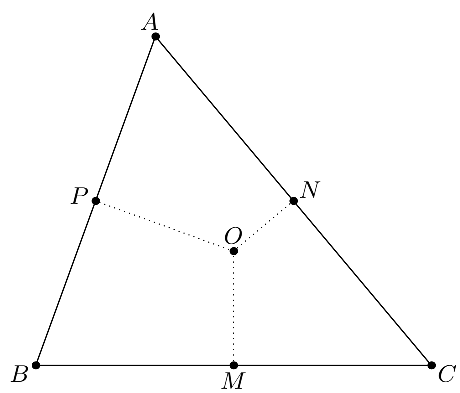

Figure 1: Carnot's Theorem

Here, OM, ON, and OP are signed lengths such that a length is positive if it intersects the interior of the triangle and negative otherwise.

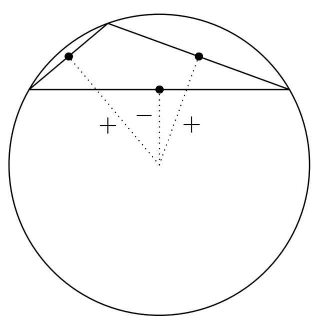

Figure 2: Directed Lengths in Carnot's Theorem

We will only give the proof for when $\triangle ABC$ is acute (and consequently all the lenghts are positive). The case when $\triangle ABC$ is obtuse is similar. In what follows, $[ABC]$ denotes the area of $\triangle ABC$.

Proof. Recall that $r(a+b+c) = 2[ABC]$. Furthermore, $2[ABC] = 2([OBC] + [OCA] + [OAB]) = a \cdot OM + b \cdot ON + c \cdot OP$. Therefore,

$$a \cdot OM + b \cdot ON + c \cdot OP = r(a+b+c). \tag{1}$$

This is starting to take form, but we need to make some trickier observations. Let

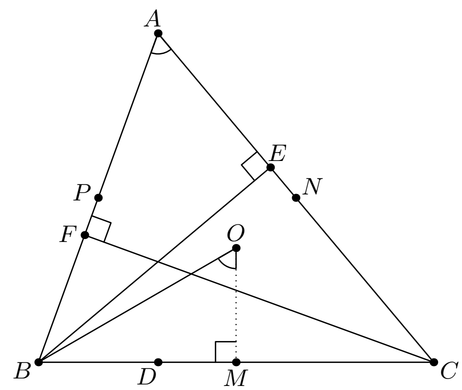

Figure 3: Proof of Carnot's Theorem

AD, BE, and CF be altitudes. Then we can find that

$$\triangle AEB \sim \triangle AFC \sim \triangle OMB$$

(note that $\angle EAB = \angle FAC = \angle MOB = \angle A$ and each triangle is right). From these similarities, we attain

$$\frac{AF}{b} = \frac{AE}{c} = \frac{OM}{R}.$$

So $OM \cdot b = R \cdot AF$ and $OM \cdot c = R \cdot AE$, implying

$$OM(b+c) = R(AE + AF).$$

We analagously can find that

$$\begin{aligned} ON(c+a) &= R(BF + BD) \\ OP(a+b) &= R(CD + CE) \end{aligned}$$

Summing these three relations, we obtain

$$\begin{aligned} OM(b+c) + ON(c+a) + OP(a+b) &= R(AE + AF + BF + BD + CD + CE) \\ &= R(a+b+c) \end{aligned}$$

Finally, summing with Equation 1, we have

$$\begin{aligned} OM(a+b+c) + ON(a+b+c) + OP(a+b+c) &= (R+r)(a+b+c) \\ OM + ON + OP &= R + r \end{aligned}$$

as desired. $\square$

A marvelous application of Carnot's theorem is the famous Japanese theorem.

Theorem 2 (Japanese Theorem)**.** Consider any cyclic polygon $P_1 P_2 \cdots P_n$. Then we can triangulate it into triangles $T_1, T_2, \cdots, T_{n-2}$ with inradii $r_1, r_2, \cdots r_{n-2}$. Then regardless of the triangulation, the sum

$$r_1 + r_2 + \cdots + r_{n-2}$$

is the same.

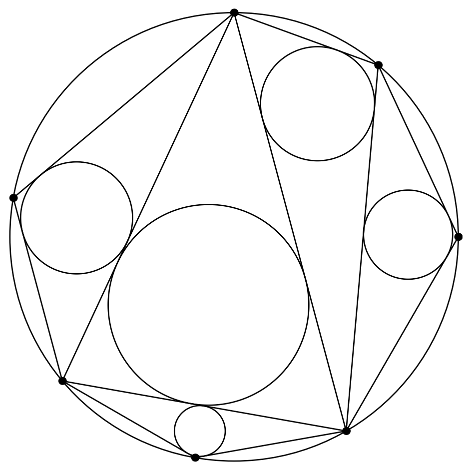

Figure 4: The Japanese Theorem

Proof. First note that each of the triangles has a common circumradius. Let the center of the circle be O. Applying Carnot's theorem to T_i, we have $r_i + R = S_i$ where S_i is the sum of the signed distances from O to the the sides of T_i. Therefore,

$$r_1 + r_2 + \cdots + r_{n-2} = S_1 + S_2 + \cdots + S_{n-2} - (n-2)R.$$

So it suffices to show that $S_1 + S_2 + \cdots S_{n-2}$ is fixed. This sum is made of perpendiculars from O to each of the sides once and each of the diagonals twice (since each diagonal is part of two different triangles). However, the distance from O to a diagonal is actually counted once with positive sign and once with negative sign, therefore cancelling out! So the sum is actually just equal to the sum of the distances from O to the sides of the polygon, which is independent of the triangulation, as desired. $\square$