The Hagge Circle

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In this note, I share one of my favorite results, the Hagge circle, which was first shown to me by Kapil Chandran. The proof follows from one nice observation after the other. First a useful definition.

Definition 1 (circumcevian triangle)**.** Consider $\triangle ABC$ and a point P. Let AP, BP, and CP intersect the circumcircle of $\triangle ABC$ at A_1, B_1, and C_1. Then we call $\triangle A_1B_1C_1$ the **circumcevian triangle** of P w.r.t. $\triangle ABC$.

Then we can state our central result.

Theorem 2 (Hagge Circle)**.** Let $\triangle A_1B_1C_1$ be the circumcevian triangle of a point P w.r.t. $\triangle ABC$. Let A', B', and C' be the reflections of A_1, B_1, and C_1 across sides BC, CA, and AB respectively. Then A', B', C', and H are concyclic, where H is the orthocenter of $\triangle ABC$. We call this circle the **Hagge circle** of P w.r.t. $\triangle ABC$.

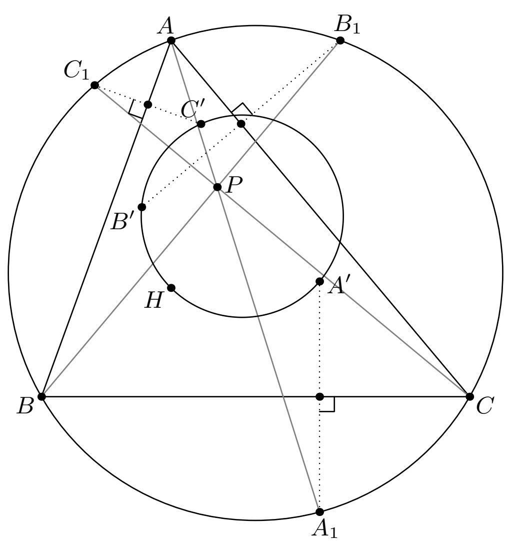

Figure 1: The Hagge Circle of P w.r.t. $\triangle ABC$

Proof. We first make the following key observation. Let Q be the isogonal conjugate of P w.r.t. $\triangle ABC$, and let $A_2B_2C_2$ be its circumcevian triangle. Then A', B', and C' are the reflections of A_2, B_2, and C_2 across the midpoints of sides BC, CA, and AB respectively. This allows us to define the Hagge in terms of the isogonal conjugate Q, as illustrated in Figure 2.

This turns out be the right way to frame the configuration. Let us now focus on the $A-$side of the diagram, which is shown in Figure 4. Let M be the midpoint of BC. Then

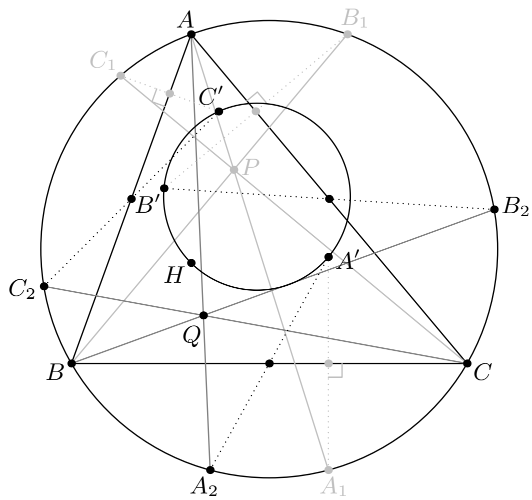

Figure 2: The Hagge Circle w.r.t. the isogonal conjugate

M is also the midpoint of $A'A_2$. By virtue of the centroid being two-thirds between a vertex and midpoint, the centroid G of $\triangle ABC$ is also the centroid of $\triangle AA'A_2$. Then the homothety $\mathcal{H}(G, -\frac{1}{2}$ sends A' to the midpoint of AA_2 and H to the circumcenter O of $\triangle ABC$.

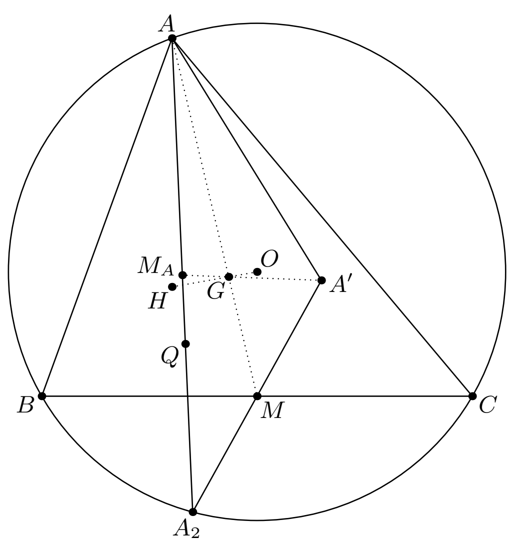

Figure 3: The $A-$side of the Hagge Circle

Now notice that the same transformation also sends B' and C' to the midpoints M_B and M_C of BB_2 and CC_2 respectively. Then instead of showing that A', B', C', and H are concyclic, we can show that M_A, M_B, M_C, and O are concyclic.

We claim that M_A, M_B, and M_C all lie on the circle with diameter OQ. Indeed, since

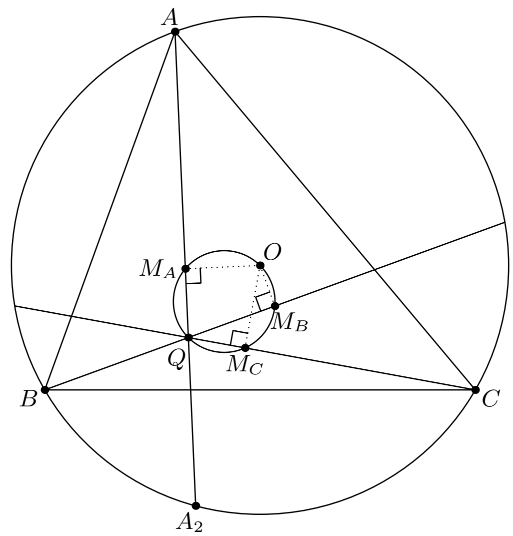

Figure 4: The Hagge Circle after $\mathcal{H}(G, -\frac{1}{2})$

M_A is the midpoint of a chord AA_2, we know that $OM_A \perp AA_2$, so $OM_AQ = 90°$. Similarly, we can show that M_B and M_C lie on the circle. □

After framing the configuration in terms of the isogonal conjugate and applying a natural homothety centered at the centroid, we arrive at much simpler concyclity. Moreover, this process tells us a little more about the Hagge circle.

Definition 3 (anticomplement)**.** Let $\triangle ABC$ have centroid G. Let P' be the image of P under $\mathcal{H}(G, -2)$. Then P' is the **anticomplement** of P w.r.t. $\triangle ABC$.

Corollary 4. Let P' be the anticomplement of the isogonal conjugate of P in $\triangle ABC$. Then HP' is the diameter of the Hagge circle of P w.r.t. $\triangle ABC$.

We now consider the special case when we choose $P = I$, the incenter. The Hagge circle of I is called the **Fuhrmann circle**.

Proposition 5 (Properties of the Fuhrmann Circle)**.** Let ω be the Fuhrmann circle of $\triangle ABC$, with center Fu. Then

(i) HNa is the diameter of ω where Na is the Nagel point of $\triangle ABC$.

(ii) $IO \parallel HNa$.

(iii) ω has radius IO.

(iv) The nine-point center N is the centroid of parallelogram $OIHFu$.

(v) The Spieker center Sp is the centroid of parallelogram $OIFuNa$.

These properties follow more or less directly from properties of points on the Euler line and the Nagel line.

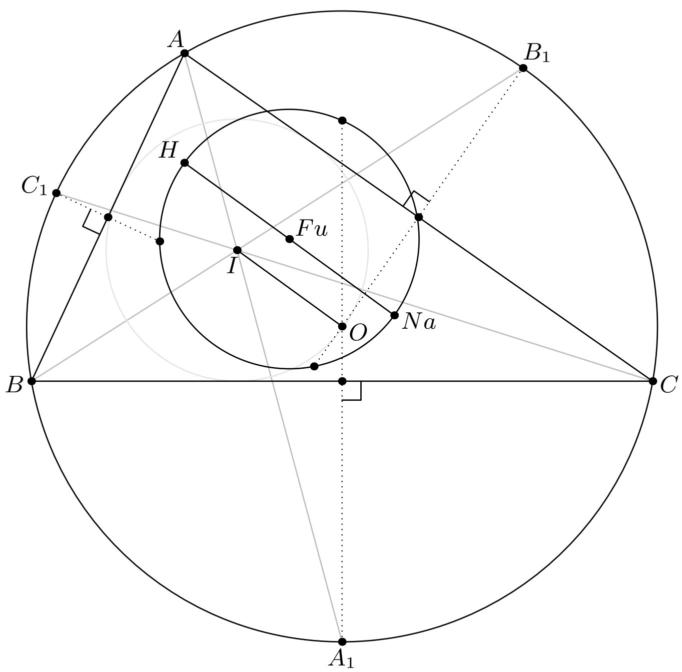

Figure 5: The Fuhrmann Circle

Proof. I is its own isogonal conjugate. Its anticomplement is the Nagel point Na. Then (i) follows from Corollary 4. We also know that $\mathcal{H}(G, -2)$ sends IO to NaH, implying (ii). This also tells us that the diameter has length $NaH = 2IO$, so the radius of ω is IO, showing (iii). (iv) follows from nine-point center being the midpoint of OH and (v) follows from the Spieker center being the midpoint of INa. □

The Fuhrmann circle ties together the numerous results involving homothety about the centroid.

References

[1] Weisstein, Eric W. "Fuhrmann Circle." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/FuhrmannCircle.html